On q-skew Iterated Ore Extensions Satisfying a Polynomial Identity

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Abstract

For iterated Ore extensions satisfying a polynomial identity we present an elementary way of erasing derivations. As a consequence we recover some results obtained by Haynal in [5]. We also prove, under mild assumptions on $R_n = R[x_1; \sigma_1, \delta_1] \ldots [x_n; \sigma_n; \delta_n]$ that the Ore extension $R[x_1; \sigma_1] \ldots [x_n; \sigma_n]$ exists and is PI if R_n is PI.

For an Ore extension $R[x; \sigma, \delta]$, where σ is an injective endomorphism of a prime ring R and δ is a σ-derivation of R, necessary and sufficient conditions for being a PI ring can be found in [7]. The main result of [5] states that if R is a noetherian domain which is also an algebra over a field k then, under some quantum like hypotheses, the iterated Ore extensions $R_n = R[x_1; \sigma_1, \delta_1] \ldots [x_n; \sigma_n, \delta_n]$ and $T_n = R[x_1; \sigma_1] \ldots [x_n; \sigma_n]$ have the same PI degree. This result was known earlier only in the case char $k = 0$ (Cf. [6]). Haynal achieved the above mentioned result by generalizing Cauchon's erasing of derivations procedure (Cf. [1]).

The aim of this paper is to present a short and elementary proof of an erasing of derivations process under the hypotheses that the iterated Ore extension is prime PI. Using this method we can recover some of Haynal's results in a slightly more general setting (Cf. Theorems 6 and 7). We believe that our approach explains also the nature of some of the assumptions.

Notice that for a general Ore extension R_n the extension T_n mentioned above does not have to exist; for example the automorphism σ_2 of R_1 has, in general, no meaning as a map of T_1. We show in Theorem 10 that, under mild assumptions on R_n, T_n does exist. Moreover it satisfies a polynomial identity provided R_n does.

Throughout the paper R stands for an associative ring with unity and $Z(R)$ for its center. For an automorphism σ and a σ-derivation δ of R, $R[x;\sigma,\delta]$ denotes the Ore extension with coefficients written on the left. We denote $R[x;\sigma]$ for $R[x;\sigma,0]$ and $R[x;\delta]$ for $R[x;id,\delta]$.

Let T be a subring of a ring W and σ, δ be an automorphism and a σ-derivation of T, respectively. We will write $T[y;\sigma,\delta] \subseteq W$ for some $y \in W$, if the set $\{y^n\}_{n=0}^{\infty}$ is left T-independent and $yr = \sigma(r)y + \delta(r)$, for any $r \in T$. That is the subring of W generated by T and y forms an Ore extension.

Let $q \in Z(R)$ be such that $\sigma(q) = q$ and $\delta(q) = 0$. We say that δ is a q-skew σ-derivation of R if $\delta\sigma = q\sigma\delta$.

We begin with the following key observation which will allow us to erase skew derivations for certain Ore extensions.

Lemma 1. *Let δ be a q-skew σ-derivation of a finite dimensional central simple algebra Q, where $q \neq 1$. Then there exists an element $y = x - b \in Q[x;\sigma,\delta]$, for a suitably chosen $b \in Q$, such that $Q[x;\sigma,\delta] = Q[y;\sigma]$.*

Proof. Assume that σ is the identity on $Z(Q)$. Then, by the Skolem-Noether's theorem, σ is an inner automorphism of Q. Let $a \in Q$ induce σ, i.e. $\sigma(r) = ara^{-1}$, for all $r \in Q$. One can easily check that $a^{-1}\delta$ is a derivation of Q and $Q[x;\sigma,\delta] = Q[a^{-1}x;a^{-1}\delta]$. Let $z \in Z(Q)$. The equalities $\delta(az) = \delta(za)$ and $\sigma(z) = z$ easily lead to $\delta(z)a = a\delta(z)$, i.e. $\delta(z) = \sigma\delta(z)$. Since $\sigma(z) = z$, we also have $\delta(z) = \delta\sigma(z) = q\sigma\delta(z)$. Hence $(1-q)\sigma\delta(z) = 0$, for $z \in Z(Q)$. By assumption $q \neq 1$, so the last equality implies that the derivation $a^{-1}\delta$ is zero on $Z(Q)$. Thus, by the Skolem-Noether's theorem, $a^{-1}\delta$ is an inner derivation of Q. Let the element $v \in Q$ be such that $a^{-1}\delta(r) = vr - rv$, for all $r \in Q$. Using the above we have $Q[x;\sigma,\delta] = Q[a^{-1}x;a^{-1}\delta] = Q[a^{-1}x - v] = Q[x - av;\sigma]$. This gives the desired conclusion in the case $\sigma|_{Z(Q)} = \mathrm{id}_{Z(Q)}$.

Assume now that there exists $c \in Z(Q)$ such that $u = \sigma(c) - c \neq 0$. Then a classical and easy computation shows that $Q[x;\sigma,\delta] = Q[x + u^{-1}\delta(c);\sigma]$. This completes the proof of the lemma. $\square$

For a prime right Goldie ring R, we denote by $Q(R)$ the classical right quotient ring of R. Let us recall that if R is a prime PI ring then, by Theorems of Posner and Kaplansky, R is right Goldie, $Q(R)$ is a central localization of R and $Q(R)$ is a finite dimensional central simple algebra (Cf. [10]).

In the following lemma we collect known results which will be used later on.

Lemma 2. *Let σ, δ be an automorphism and a σ-derivation of a ring R, respectively. Then:*

(1) *Let $\mathcal{S} \subseteq R$ be a right Ore set of regular elements of R such that $\sigma(\mathcal{S}) \subseteq \mathcal{S}$. Then:*

   (a) *σ and δ have unique extensions to an automorphism and a σ-derivation of $R\mathcal{S}^{-1}$, respectively. Moreover if δ is a q-skew σ-derivation of R, then the extension of δ to $R\mathcal{S}^{-1}$ is also a q-skew σ-derivation of $\mathcal{R}S^{-1}$;*

(b) $\mathcal{S}$ *is a right Ore set of regular elements of* $R[x;\sigma,\delta]$ *and the localization* $(R[x;\sigma,\delta])\mathcal{S}^{-1}$ *is isomorphic to* $(R\mathcal{S}^{-1})[x;\sigma,\delta]$.

(2) *Suppose* R *is a prime* PI *ring. Then:*

(a) $R[x;\sigma,\delta]$ *is a PI ring if and only if* $Q(R)[x;\sigma,\delta]$ *is a PI ring;*

(b) *If* $R[x;\sigma,\delta]$ *is a PI ring, then* $Q(R)[x;\sigma,\delta] \subseteq Q(R[x;\sigma,\delta])$, *and* $Q(R[x;\sigma,\delta])$ *is isomorphic to* $Q(Q(R)[x;\sigma,\delta])$.

Proof. The statement (1) is part of folklore and we refer to Lemmas 1.3 and 1.4 of [4] for its proof.

The statement (2)(a) is exactly Proposition 1.6 of [7].

(2)(b) Suppose $R[x;\sigma,\delta]$ is a PI ring. Notice that $R[x;\sigma,\delta]$ is a prime ring as R is such and the statement (2)(a) implies that all rings appearing in (2)(b) are prime right Goldie rings. Moreover every regular element of R is regular in $R[x;\sigma,\delta]$, as σ is an automorphism of R. Now the thesis is a direct consequence of (1) and universal properties of localizations. $\square$

Remark 3. It is known (Cf. [8]) that if R is a prime (semiprime) right Goldie ring, then so is $R[x;\sigma,\delta]$. Thus all rings appearing in Lemma 2(2)(b) are prime (semiprime) Goldie. This implies that $Q(R)[x;\sigma,\delta] \subseteq Q(R[x;\sigma,\delta]) \simeq Q(Q(R)[x;\sigma,\delta])$, for general Ore extension over prime (semiprime) Goldie rings.

Theorem 4. *Suppose that* δ *is a q-skew σ-derivation of a prime ring* R, *where* $q \neq 1$. *Then:*

(1) *The following conditions are equivalent:*

(a) $R[x;\sigma,\delta]$ *is a PI ring;*

(b) $R[x;\sigma]$ *is a PI ring;*

(c) R *is a PI ring and the restriction of* σ *to the center* $Z(R)$ *of* R *is an automorphism of finite order.*

(2) *Suppose that one of the equivalent conditions of (1) holds, then:*

(a) *The quotient rings* $Q(R[x;\sigma,\delta])$ *and* $Q(R[x;\sigma])$ *are isomorphic. In particular, the* PI-*degrees of* $R[x;\sigma,\delta]$ *and* $R[x;\sigma]$ *are equal;*

(b) *there exists an element* $y \in R[x;\sigma,\delta]$ *such that* $R[y;\sigma] \subseteq R[x;\sigma,\delta]$.

Proof. (1) Suppose that one of the rings $R[x;\sigma]$ or $R[x;\sigma,\delta]$ is PI. Then R is prime PI and Lemma 2(2)(a) shows that, replacing R by $Q(R)$, we may assume that R is a central simple algebra finite dimensional over its center. In this case, Lemma 1 implies that $R[x;\sigma] \simeq R[x;\sigma,\delta]$, this gives the equivalence between statements (a) and (b).

The equivalence $(b) \Leftrightarrow (c)$ is part of Proposition 2.5 in [7].

(2)(a) By assumption both $R[x; \sigma, \delta]$ and $R[x; \sigma]$ are PI rings. Then, by Lemma 2(2)(b) and Lemma 1, we have: $Q(R[x; \sigma, \delta]) \simeq Q(Q(R)[x; \sigma, \delta]) \simeq Q(Q(R)[x; \sigma]) \simeq Q(R[x; \sigma])$. This gives the first statement.

Let us recall that the PI degree of a prime PI ring is also the PI degree of its classical ring of quotients. This observation completes the proof of (2).

(2)(b) By assumption $R[x; \sigma, \delta]$ is a PI ring, then so is $Q(R)$ and Lemma 1 shows that there exists $b \in Q(R)$ such that $Q(R)[x; \sigma, \delta] = Q(R)[y'; \sigma]$, where $y' = x - b$. By the theorem of Posner, we can pick $0 \neq c \in Z(R)$ such that $y := cy' = cx - a \in R[x; \sigma, \delta]$. Thus, as $y'r = \sigma(r)y'$, we also have $yr = \sigma(r)y$, for all $r \in R$. Notice that $y^n = c\sigma(c) \dots \sigma^{n-1}(c)y'^n$. Therefore the set $\{y^n\}_{n=0}^{\infty}$, is left R-independent. This shows that $R[y; \sigma] \subseteq R[x; \sigma, \delta]$ and completes the proof. $\qquad \square$

Since an Ore extension of a prime ring is again a prime ring, we can use the above Theorem 4 to erase derivations appearing in a PI iterated Ore extension as long as the derivations are quantized. This erasing of derivations process under the assumption of PI doesn't assume that the derivations are locally nilpotent nor that the prime base ring is of zero characteristic. In this sense it gives a generalization of the well known Cauchon's process of erasing derivations (Cf. [1]). However some care is needed. Notice that even in the case of iterated Ore extension $R[x_1; \sigma_1.\delta_1][x_2; \sigma_2.\delta_2]$ of length two, the iterated extension $R[x_1; \sigma_1][x_2; \sigma_2]$ has no meaning in general, as the automorphism σ_2 is defined on the ring $R[x_1; \sigma_1.\delta_1]$ but not on $R[x_1; \sigma_1]$. The following lemma contains observations which both explain the assumptions made in Theorem 6 and show that the iterated Ore extensions of automorphism type in this theorem do exist. First let us consider a simple example. Let k be a field and $W = k\langle x_1, x_2 \mid x_2x_1 = \lambda x_1x_2 \rangle$ where $0 \neq \lambda \in k$. Notice that W can be presented either as $k[x_1][x_2; \sigma]$ or $k[x_2][x_1; \tau]$, where σ and τ are k-automorphisms of appropriate polynomial rings defined by $\sigma(x_1) = \lambda x_1$, and $\tau(x_2) = \lambda^{-1}x_2$. The statement (2) of the following lemma generalizes this observation to Ore extensions.

Lemma 5. *Suppose that σ_1, σ_2 are automorphisms and δ_1 is a σ_1-derivation of a ring R. Let $\lambda \in R$ be an invertible element. Then:*

(1) *σ_2 can be extended to an automorphism of $R[x_1; \sigma_1, \delta_1]$ by setting $\sigma_2(x_1) = \lambda x_1$ if and only if $\sigma_2\sigma_1(r) = \lambda\sigma_1\sigma_2(r)\lambda^{-1}$ and $\sigma_2\delta_1(r) = \lambda\delta_1\sigma_2(r)$, for any $r \in R$;*

(2) *Suppose σ_2 has been extended to $R[x_1; \sigma_1, \delta_1]$ as above. Then there exist an automorphism σ_1' and a σ_1'-derivation δ_1' of $R[x_2; \sigma_2]$ such that $R[x_1; \sigma_1, \delta_1][x_2; \sigma_2] = R[x_2; \sigma_2][x_1; \sigma_1', \delta_1']$ where $\sigma_1'|_R = \sigma_1$, $\sigma_1'(x_2) = \lambda^{-1}x_2$ and $\delta_1'|_R = \delta_1$, $\delta_1'(x_2) = 0$;*

(3) *With the same notation as in (2) above, suppose additionally that δ_1 is a q-quantized σ_1-derivation. Then δ_1' is also a q-quantized σ_1'-derivation of $R[x_2; \sigma_2]$ if and only if $\delta_1(\lambda) = 0$.*

Proof. A standard proof is left to the reader. $\qquad \square$

Let us make some remarks about the hypotheses that will appear in the next theorem. In view of Lemma 5(1) it will be natural, to avoid technicalities, to assume that λ_{ij}'s from Theorem 6 are central in R_n. This means, in particular, that the λ_{ij}'s are also fixed by appearing automorphisms σ_i's. This, in turn, guarantees the existence of the extension T_n from the theorem.

The method of switching indeterminates in the proof of Theorem 6, based on Lemma 5(2), goes back to the paper [5]. The statement (3) from the above lemma explains that, in the process of switching indeterminates in Theorem 6, we need all the λ_{ij}'s to be also invariant under the action of all suitable skew derivations. A more general situation will be considered in Proposition 10.

Theorem 6. *Suppose $R = R_0$ is a prime ring and $n \geq 1$. Let $R_i := R_{i-1}[x_i; \sigma_i, \delta_i]$, $1 \leq i \leq n$, be a sequence of Ore extensions such that:*

(i) For any $1 \leq i \leq n$ and $1 \leq j < i \leq n$:
- *$\sigma_i|_{R_0}$ is an automorphism of R_0;*
- *$\sigma_i(x_j) = \lambda_{ij} x_j$ where $\lambda_{ij} \in Z(R_0)$ is a unit such that $\sigma_k(\lambda_{ij}) = \lambda_{ij}$, $\delta_k(\lambda_{ij}) = 0$, for all $i \leq k \leq n$;*

(ii) For any $1 \leq i \leq n$, δ_i is a q_i-skew σ_i-derivation of R_{i-1}, where $q_i \neq 1$.

Then the following conditions are equivalent:

(1) R_n is a PI ring.

(2) There exist elements $y_i \in R_n$, where $1 \leq i \leq n$, such that R_n contains a subring T_n, which is an Ore extension of the form $T_n := R_0[y_1; \tau_1][y_2; \tau_2]\ldots[y_n; \tau_n]$ where $\tau_i|_{R_0} = \sigma_i|_{R_0}$, for $1 \leq i \leq n$ and $\tau_i(y_j) = \lambda_{ij} y_j$, for $1 \leq j < i \leq n$. Moreover T_n is PI and the quotient rings $Q(T_n)$ and $Q(R_n)$ are isomorphic.

Proof. Notice first that all rings considered in the theorem are prime, as $R = R_0$ is prime. Observe also that $\sigma_i(R_j) = R_j$, for $0 \leq j < i \leq n$.

$(1) \Rightarrow (2)$ We proceed by induction on n. The case $n = 1$ is a direct consequence of Theorem 4.

Suppose now that $n > 1$ and R_n is a PI ring. Theorem 4 shows that there exists an element $y_n \in R_n$ such that the subring $T = R_{n-1}[y_n; \sigma_n] \subseteq R_n$ generated by R_{n-1} and y_n satisfies $Q(T) \simeq Q(R_n)$.

Using Lemma 5(2) we can write $T = R_{n-2}[y_n; \sigma_n][x_{n-1}; \sigma'_{n-1}, \delta'_{n-1}]$, where σ'_{n-1} is given by $\sigma'_{n-1}|_{R_{n-2}} = \sigma_{n-1}|_{R_{n-2}}$, $\sigma'_{n-1}(y_n) = \lambda_{n,n-1}^{-1} y_n$ and δ'_{n-1} is the extension of $\delta_{n-1}|_{R_{n-2}}$ to $R_{n-2}[y_n; \sigma_n]$ obtained by setting $\delta'_{n-1}(y_n) = 0$. Continuing this process we can present T in the following way:

$$T = R_0[y_n; \sigma_n][x_1; \sigma'_1, \delta'_1]\ldots[x_{n-1}; \sigma'_{n-1}, \delta'_{n-1}],$$

where $\sigma'_i(y_n) = \lambda_{ni}^{-1} y_n$ and $\sigma'_i(x_j) = \lambda_{ij} x_j$, for all $1 \leq i \leq n-1$ and $1 \leq j < i \leq n-1$. Lemma 5(3) shows that the σ'_i-derivations δ'_i remain q_i-quantized, $1 \leq i \leq n-1$. We can now apply the induction hypothesis to T, replacing R_0 by $R_0[y_n; \sigma_n]$, and conclude that there exist elements $y_1, \ldots, y_{n-1} \in T$ such that the subring T_n generated by $R_0[y_n; \sigma_n]$ and $y_1, \ldots, y_{n-1}$ satisfies

$$T_n := R_0[y_n; \sigma_n][y_1; \sigma''_1] \ldots [y_{n-1}; \sigma''_{n-1}] \subseteq T \text{ and } Q(T_n) \simeq Q(T)$$

where $\sigma''_i|_{R_0} = \sigma'_i|_{R_0} = \sigma_i|_{R_0}$, $\sigma''_i(y_n) = \sigma'_i(y_n) = \lambda_{ni}^{-1} y_n$, $\sigma''_i(y_j) = \sigma_i(y_j) = \lambda_{ij} y_j$, for all $1 \leq i \leq n-1$ and $1 \leq j < i \leq n-1$.

Now, using Lemma 5(2) again, we can reorder the $y'_i s$ to get that $T_n = R_0[y_1; \tau_1][y_2; \tau_2] \ldots [y_n; \tau_n]$, where $\tau_i|_{R_0} = \sigma_i|_{R_0}$ and $\tau_i(y_j) = \lambda_{ij} y_j$, for any $1 \leq j < i \leq n$. Therefore

$$R_0[y_1; \tau_1][y_2; \tau_2] \ldots [y_n; \tau_n] = T_n \subseteq T \subseteq R_n$$

as required. As it was proved above we also have $Q(R_n) \simeq Q(T) \simeq Q(T_n)$. Clearly T_n is a PI ring as a subring of the PI ring R_n. This completes the proof of $(1) \Rightarrow (2)$.

The implication $(2) \Rightarrow (1)$ is clear. $\square$

The above theorem is a partial generalization of the main result of [5]. Comparing Theorem 6 above with Theorem 4.6 [5], observe that in [5] it is additionally assumed that each δ_i, with $1 \leq i \leq n$, extends to locally nilpotent iterative higher q_i-skew σ_i-derivation on R_{i-1} (see [5] for details). Notice also that in [5], R is a noetherian domain which is an algebra over a field k and $q_i, \lambda_{ij} \in k$, where $q_i \notin \{0, 1\}$. This is due to the fact that higher q-skew derivations where used in [5] for erasing derivations from Ore extensions. The assumption that $\sigma_i|_{R_0}$ is an automorphism of R_0, for all $1 \leq i \leq n$, from Theorem 6 was not formally stated in Theorem 4.6 [5] but it was used in its proof.

Most of the quantum algebras can be presented as iterated Ore extensions of the form $k[x_1][x_2; \sigma_2, \delta_2] \ldots [x_n; \sigma_n, \delta_n]$, where k is a field and the appearing automorphisms and skew derivations are as in Theorem 6. Thus we record the following theorem which covers this case by taking $R_0 = k$, $\sigma_1 = \mathrm{id}_k$, $\delta_1 = 0$ and $q_1 = 0$.

To make the presentation a bit shorter, we formulate the theorem in the language of algebras. If the Ore extension $R[x; \sigma, \delta]$ is PI, then so is the ring R, thus we will assume that R satisfies a polynomial identity.

Theorem 7. *Suppose that $R = R_0$ is a prime PI algebra over a field k and $n \geq 1$. Let $R_i := R_{i-1}[x_i; \sigma_i, \delta_i]$, $1 \leq i \leq n$, be a sequence of Ore extensions such that each σ_i is a k-linear automorphism of R_{i-1} and each δ_i is a k-linear σ_i-derivation of R_{i-1} such that:*

(i) $\sigma_i|_{R_0}$ is an automorphism of R_0 of finite order, for any $1 \leq i \leq n$;

(ii) $\sigma_i(x_j) = \lambda_{ij} x_j$ where $0 \neq \lambda_{ij} \in k$, for any $1 \leq j < i \leq n$;

(iii) δ_i is a q_i-skew σ_i-derivation of R_{i-1}, where $1 \neq q_i \in k$, for any $1 \leq i \leq n$.

Then the following conditions are equivalent:

(1) R_n *is a* PI *algebra;*

(2) $T_n = R_0[y_1; \sigma'_1][y_2; \sigma'_2] \cdots [y_n; \sigma'_n]$ *is a* PI *algebra where* $\sigma'_i|_{R_0} = \sigma_i|_{R_0}$ *and* $\sigma'_i(y_j) = \lambda_{ij} y_j$, *for* $1 \leq i \leq n$ *and* $1 \leq j < i \leq n$;

(3) λ_{ij} *is a root of unity, for any* $1 \leq j < i \leq n$;

(4) σ_i *is an automorphism of finite order of* R_{i-1}, *for any* $1 \leq i \leq n$.

Moreover if one of the above equivalent conditions holds, then the algebras R_n *and* T_n *have isomorphic classical rings of quotients and equal* PI *degrees.*

Proof. The implication (1) $\Rightarrow$ (2) and the additional statements are direct consequences of Theorem 6.

(2) $\Rightarrow$ (3) Let us fix $1 \leq j < i \leq n$. Since $\sigma'_j|_{R_0} = \sigma_j|_{R_0}$, the assumption (i) gives an $0 \neq n_j \in \mathbb{N}$ such that $(\sigma'_j|_{R_0})^{n_j} = \mathrm{id}_{R_0}$. This means that $y_j^{n_j} \in Z(R_0[y_j; \sigma'_j])$.

Now let us consider the subring T_{ji} of T_n generated by R_0, y_j, y_i. The assumptions imposed imply that T_{ji} is the Ore extension $R_0[y_j; \sigma'_j][y_i; \sigma'_i]$ which is PI as a subalgebra of T_n. Hence, by Theorem 4(1), σ'_i is of finite order on $Z(R_0[y_i; \sigma'_i])$. In particular there exists $0 \neq l_i \in \mathbb{N}$ such that $\sigma'^{l_i}_i(y_j^{n_j}) = y_j^{n_j}$. This leads to $\lambda_{ij}^{n_j l_i} = 1$, as required.

(3) $\Rightarrow$ (4) Fix $1 \leq i \leq n$. By assumption, we can find $m \geq 1$ such that $(\sigma_i|_{R_0})^m = \mathrm{id}_{R_0}$ and $\lambda_{ij}^m = 1$, for all $1 \leq j < i \leq n$. This implies that the automorphism σ_i is of finite order on R_{i-1}.

(4) $\Rightarrow$ (1) We will proceed by induction on n. For $n = 1$, the implication is a direct consequence of Theorem 4(1).

Suppose $n > 1$. Recall that, by assumption, R_0 is a prime PI algebra. The case $n = 1$, treated above, shows that $R_1 = R_0[x_1; \sigma_1, \delta_1]$ is a prime PI algebra as well. Thus, the set $\mathcal{S}$ of all regular elements of R_1 is a right Ore set and, using Lemma 2(1)(b) repeatedly, we see that $\mathcal{S}$ is a right Ore set of regular elements of R_n and $R_n \mathcal{S}^{-1}$ is isomorphic to $Q(R_1)[x_2; \sigma_2, \delta_2] \ldots [x_n; \sigma_n, \delta_n]$.

Notice that, due to the assumption (4), the extensions of the automorphisms σ_i to $Q(R_1)$, $2 \leq i \leq n$, possess the property (i) with respect to $Q(R_1)$ and clearly also satisfy (ii). Moreover Lemma 2(1)(a) implies that the extensions of σ_i-derivations δ_i to $Q(R_1)$ remain q_i-quantized, $2 \leq i \leq n$, i.e. (ii) holds. Therefore, applying the induction hypothesis to $Q(R_1)[x_2; \sigma_2, \delta_2] \ldots [x_n; \sigma_n, \delta_n] \simeq R_n \mathcal{S}^{-1}$ we can conclude that $R_n \mathcal{S}^{-1}$ satisfies polynomial identity and so does R_n. $\square$

Both Theorem 1.2 (1) and Corollary 4.7 of [5] are direct consequences of the above theorem. Moreover we relaxed the assumptions from [5] that δ_i's have to extend to locally nilpotent iterative higher q_i-skew σ_i-derivations and that R is a noetherian domain.

Let us mention that Haynal applied Corollary 4.7 [5] and a result of De Concini and Procesi (Cf. [3]) to compute PI degrees of some quantum algebras.

Notice that the Weyl algebra over a field k of characteristic 0 does not satisfy a polynomial identity but $k[x]$ does. This shows that the implication $(2) \Rightarrow (1)$ of Theorem 7 does not hold if we allow one of the q_i's to be equal to one. On the other hand, we will show in Theorem 10, that the implication $(1) \Rightarrow (2)$ of Theorem 7 holds even when skew derivations δ_i's are not quantized and the base ring R is not prime. The example of Weyl algebra over a field k of characteristic $p \neq 0$ shows that the PI degrees of R_n and T_n can be different in this case.

For the proof of Theorem 10 some preparation is needed. Let σ be an automorphism of the $\mathbb{N}$-graded ring $B = \bigoplus_{i=0}^{\infty} B_i$ (resp. of the filtered ring $C = \bigcup_{i=0}^{\infty} C_i$), we say that σ respects the gradation (resp. the filtration) if $\sigma(B_i) = B_i$ (resp. $\sigma(C_i) = C_i$), for all $i \geq 0$. The associated graded ring $C_0 \oplus \bigoplus_{i=1}^{\infty} (C_i/C_{i-1})$ of the filtered ring $C = \bigcup_{i=0}^{\infty} C_i$ will be denoted by $\mathfrak{gr}(C)$.

Henceforward we will extend slightly our previous notation and write also $A[x; \sigma]$ for the additive group of all polynomials from skew polynomial ring $R[x; \sigma]$ consisting of all polynomials with coefficients in an additive group A of R.

Lemma 8. *Suppose that $C = \bigcup_{i=0}^{\infty} C_i$ is a filtered ring and σ is an automorphism of C which respects the filtration. Then:*

(1) *σ induces an automorphism $\overline{\sigma}$ of the associated graded ring $\mathfrak{gr}(C)$ which respects the gradation of $\mathfrak{gr}(C)$. Moreover $\overline{\sigma}|_{C_0} = \sigma|_{C_0}$.*

(2) *The ring $B = C[x; \sigma]$ has a natural filtration $B = \bigcup_{i=0}^{\infty} B_i$, where $B_i = C_i[x; \sigma]$. The associated graded ring $\mathfrak{gr}(B)$ is isomorphic to $\mathfrak{gr}(C)[x; \overline{\sigma}]$, where $\overline{\sigma}$ denotes the automorphism defined in (1).*

(3) *Let $T_n = C[x_1; \sigma_1] \ldots [x_n; \sigma_n]$ be an iterated Ore extension such that:*
   - *For any $1 \leq i \leq n$, $\sigma_i|_C$ respects the filtration of C;*
   - *For any $1 \leq j < i \leq n$, $\sigma_i(x_j) = a_{ij} x_j$, for some invertible elements $a_{ij} \in C_0$.*
   Then $T_n = \bigcup_{i=0}^{\infty} C_i[x_1; \sigma_1] \cdots [x_n; \sigma_n]$ is a filtration of T_n such that the associated graded ring $\mathfrak{gr}(T_n)$ is isomorphic to $\mathfrak{gr}(C)[x_1; \sigma_1'] \ldots [x_n; \sigma_n']$ where the automorphisms σ_i''s satisfy:
   - *For any $1 \leq i \leq n$, $\sigma_i'|_{\mathfrak{gr}(C)} = \overline{\sigma_i|_C}$ defined as in (1);*
   - *For any $1 \leq j < i \leq n$, $\sigma_i'(x_j) = a_{ij} x_j$.*

Proof. (1) Let us extend the automorphism σ of C to an automorphism of $C[x]$ by setting $\sigma(x) = x$. Then, by the assumption, σ preserves the filtration of C. Hence σ induces an automorphism of the Rees extension $\mathfrak{R}(C) = \sum_{i=0}^{\infty} C_i x^i \subseteq C[x]$. Let $I = \mathfrak{R}(C)x$. Then $\sigma(I) = I$ and σ induces an automorphism $\overline{\sigma}$ of $\mathfrak{R}(C)/I \simeq \mathfrak{gr}(C)$ which has the desired properties.

(2) By the statement (1), $\mathfrak{gr}(C)[x; \overline{\sigma}]$ exists. The automorphism σ preserves the filtration of the base ring C. Thus it is clear that $B = \bigcup_{i=0}^{\infty} B_i$, where $B_i = C_i[x; \sigma]$ is a filtration of $C[x; \sigma]$. The isomorphism between the graded rings $\mathfrak{gr}(C[x; \sigma])$ and $\mathfrak{gr}(C)[x; \overline{\sigma}]$ is given by the natural isomorphism of homogeneous components $C_i[x; \sigma]/C_{i-1}[x; \sigma]$ and $(C_i/C_{i-1})[x; \overline{\sigma}]$, where $i \geq 0$ and $C_{-1} = 0$.

(3) We proceed by induction on n. The case $n = 1$ is given by the statement (2).

Let $n > 1$. By the induction hypothesis, we know that $T_{n-1} = \bigcup_{i=0}^{\infty} C_i[x_1; \sigma_1] \cdots [x_{n-1}; \sigma_{n-1}]$ is a filtration for T_{n-1} such that:

• the associated graded ring $\mathfrak{gr}(T_{n-1})$ is isomorphic to $\mathfrak{gr}(C)[x_1; \sigma'_1] \cdots [x_{n-1}; \sigma'_{n-1}]$ where

• $\sigma'_i|_{\mathfrak{gr}(C)} = \overline{\sigma_i|_C}$ defined as in (1), for any $1 \leq i \leq n - 1$;

• $\sigma'_i(x_j) = a_{ij}x_j$, for any $1 \leq j < i \leq n - 1$.

Let us write T_n as $T_n = T_{n-1}[x_n; \sigma_n]$. Notice that, by the assumptions imposed on σ_n, the filtration of T_{n-1} is respected by σ_n. Therefore, by (2), we can extend the filtration of T_{n-1} to T_n by setting

$$(T_n)_i = (T_{n-1})_i[x_n; \sigma_n] = C_i[x_1; \sigma_1] \ldots [x_n; \sigma_n], \text{ for all } i \geq 0.$$

Hence, making use of (2) and (1) with $C = T_{n-1}$, we obtain

$$\mathfrak{gr}(T_n) \simeq \mathfrak{gr}(T_{n-1})[x_n; \sigma'_n] \simeq \mathfrak{gr}(C)[x_1; \sigma'_1] \cdots [x_{n-1}; \sigma'_{n-1}][x_n; \sigma'_n]$$

where $\sigma'_n = \overline{\sigma_n}$, as defined in (1). In particular, $\overline{\sigma_n}|_{C_0[x_1; \sigma_1] \ldots [x_{n-1}; \sigma_{n-1}]} = \sigma_n|_{C_0[x_1; \sigma_1] \ldots [x_{n-1}; \sigma_{n-1}]}$. This shows that $\sigma'_n(x_i) = \sigma_n(x_i) = a_{ni}x_i$, for $1 \leq i \leq n - 1$. Since $\mathfrak{gr}(C) \subseteq \mathfrak{gr}(T_{n-1})$, we also have $\sigma'_n|_{\mathfrak{gr}(C)} = \overline{\sigma_n}|_{\mathfrak{gr}(C)} = \overline{\sigma_n|_C}$. This shows that the automorphism σ'_n has the desired properties and completes the proof of the lemma. $\square$

Lemma 9. *Let $n \geq 1$ and $R = R_0$. Suppose that, for $1 \leq i \leq n$, the iterated Ore extension $R_i = R_{i-1}[x_i; \sigma_i, \delta_i]$ is given and:*

(i) for any $1 \leq i \leq n$, $\sigma_i|_{R_0}$ is an automorphism of R_0;

(ii) for any $1 \leq j < i \leq n$, $\sigma_i(x_j) = a_{ij}x_j + c_{ij}$, where $a_{ij} \in R_0$ is invertible and $c_{ij} \in R_{j-1}$.

Then $\sigma_i|_{R_j}$ is an automorphism of R_j, for any $0 \leq j < i \leq n$. Moreover $\sigma_i|_{R_j}$ respects the filtration of R_j determined by the degree in x_j, when $j \geq 1$.

Proof. The inclusion $\sigma_i(R_j) \subseteq R_j$ is clear from the definition of σ_i. The reverse inclusion is obtained by induction on $0 \leq j < i$. The case $j = 0$ is given by the hypothesis (i). If $j > 0$, the induction hypothesis shows that there exist elements $b_{ij} \in R_0$, $d_{ij} \in R_{j-1}$ such that $\sigma_i(b_{ij}) = a_{ij}^{-1}$ and $\sigma_i(d_{ij}) = c_{ij}$. This gives $x_j = \sigma_i(b_{ij}x_j - d_{ij})$. In particular, $x_j \in \sigma_i(R_j)$. It is now easy to conclude that $\sigma_i|_{R_j}$ is an automorphism of R_j which respects the filtration of R_j given by the degree in x_j, when $j \geq 1$. $\square$

It was observed in Proposition 1.2 [7] that if a filtered ring C satisfies a polynomial identity, then the same is true for the associated graded ring $\mathfrak{gr}(C)$. Thus the skew polynomial ring $R[x; \sigma]$ is always a PI ring, provided $R[x; \sigma, \delta]$ is such. In the following theorem we extend this result to iterated Ore extensions.

Theorem 10. *Let $n \geq 1$ and $R = R_0$. Suppose that, for $1 \leq i \leq n$, the iterated Ore extension $R_i = R_{i-1}[x_i; \sigma_i, \delta_i]$ is given and:*

(i) for any $1 \le i \le n$, $\sigma_i|_{R_0}$ is an automorphism of R_0;

(ii) for any $1 \le j < i \le n$, $\sigma_i(x_j) = a_{ij}x_j + c_{ij}$, where $a_{ij} \in R_0$ is invertible and $c_{ij} \in R_{j-1}$.

Then there exists an iterated Ore extension $T_n = R_0[y_1; \sigma'_1] \ldots [y_n; \sigma'_n]$ such that:

(1) $\sigma'_1 = \sigma_1$, $\sigma'_i|_{R_0} = \sigma_i|_{R_0}$ and $\sigma'_i(y_j) = a_{ij}y_j$, for all $1 \le j < i \le n$;

(2) If R_n is a PI ring, then T_n is also a PI ring.

Proof. Let us first remark that, for any $0 \le j < i \le n$, $\sigma_i|_{R_j}$ is an automorphism of R_j which respects the filtration of R_j determined by the degree in x_j.

(1) We construct, by induction, a sequence $W_0, W_1, \ldots, W_n = T_n$ of filtered rings. Let us put $W_0 = R_n$. W_0 is naturally filtered by the degree in x_n and we set $W_1 = \mathfrak{gr}(W_0) \simeq R_{n-1}[y_n; \sigma_n]$. By Lemma 9, the filtration of R_{n-1} given by the degree in x_{n-1} is respected by σ_n. The filtration on W_1 is given by extending, as in Lemma 8(2), this filtration of R_{n-1}.

Suppose $1 \le s < n$ and the extension $W_s = R_{n-s}[y_{n-s+1}; \mu_{n-s+1}] \ldots [y_n; \mu_n]$ is defined, where:

$$\mu_i|_{R_{n-s}} = \sigma_i|_{R_{n-s}} \text{ and } \mu_i(y_j) = a_{ij}y_j, \text{ for any } n-s+1 \le i \le n \text{ and } n-s+1 \le j < i.$$

Now we can apply Lemma 9 and Lemma 8 with $C = R_{n-s} = R_{n-s-1}[x_{n-s}, \sigma_{n-s}, \delta_{n-s}]$ filtered by the degree in x_{n-s} to obtain a filtration on W_s such that

$$\mathfrak{gr}(W_s) \simeq \mathfrak{gr}(R_{n-s})[y_{n-s+1}; \mu'_{n-s+1}] \ldots [y_n; \mu'_n] \simeq R_{n-s-1}[y_{n-s}; \sigma_{n-s}][y_{n-s+1}; \mu'_{n-s+1}] \ldots [y_n; \mu'_n]$$

where

$$\mu'_i|_{R_{n-s-1}} = \sigma_i|_{R_{n-s-1}} \text{ and } \mu'_i(y_j) = a_{ij}y_j, \text{ for any } n-s \le i \le n \text{ and } n-s \le j < i$$

with $\mu'_{n-s} = \sigma_{n-s}$. We define W_{s+1} by setting $W_{s+1} = \mathfrak{gr}(W_s)$. The desired iterated skew polynomial ring T_n is $T_n = W_n$.

(2) Suppose that $R_n = W_0$ satisfies a polynomial identity. Notice that each W_{s+1}, with $0 \le s < n$, is defined as $W_{s+1} = \mathfrak{gr}(W_s)$. Therefore $T_n = W_n$ is a PI ring, by Proposition 1.2 of [7] which says that the associated graded ring $\mathfrak{gr}(C)$ of a filtered ring C is PI, provided C is such. □

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